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DIFFERENTIAL METHODS FOR STUDYING RADIANT HEAT TRANSFER

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Аннотация—В работе рассмотрены дифференциальные методы исследования лучистого теплообмена, которые благодаря своей сравнительной простоте открывают новые возможности. Предложен более точный дифференциальный метод, основывающийся на тензорном представлении лучистого потока. Сравнение результатов расчёта лучистого теплообмена в плоском слое поглощающей среды, полученных на основе существующих и предлагаемого методов, с численным решением задачи на вычислительной машине показало очень хорошее совпадение для рекомендуемого метода.

NOMENCLATURE

k,	absorption coefficient of the medium;
l,	length of path of radiation;
$E_0 = \sigma_0 T^4$,	black-body emissivity;
J,	radiation intensity;
$J_0 = \dfrac{\sigma_0 T^4}{\pi}$,	black-body radiation intensity;
$E_+ = \int_{(+2\pi)} J\eta \, d\omega$, $E_- = \int_{(-2\pi)} J\eta \, d\omega$	densities of hemispherical incident fluxes at a plane normal to the x-axis;
$\eta \equiv \lvert\cos(\widehat{n, s})\rvert = \cos\theta$,	absolute value of the cosine of the angle between the direction of the beam (s) and the normal (n) surface section;
$d\omega$,	element of the solid angle in the direction s;
$m_+ = \dfrac{\int_{(+2\pi)} J_+ \, d\omega}{\int_{(+2\pi)} J_+ \eta \, d\omega}$, $m_- = \dfrac{\int_{(-2\pi)} J_- \, d\omega}{\int_{(-2\pi)} J_- \eta \, d\omega}$	coefficients allowing for distribution of J in hemispherical radiative fluxes E_+ and E_-;
T,	absolute temperature of the medium in the same section of the layer;
T_1 and T_2, $E_{0\cdot1}$ and $E_{0\cdot2}$,	temperatures and black-body emissivities of boundary surfaces, respectively;
ϵ_1 and ϵ_2, A_1 and A_2,	emissivity and absorptivity of the boundary surfaces, respectively;
δ,	thickness of the plane layer;
$\mathbf{q}$,	radiative flux vector;
c,	velocity of propagation of radiation in the medium;
$U = \dfrac{1}{c} \int_{(4\pi)} J \, d\omega$,	volume density of radiation energy;
P,	symmetrical orthogonal emissive power tensor of the second range (P/c is tensor of radiation intensities);

$A = \frac{P}{cU}$, dimensionless tensor equal to the ratio of tensor P to its main invariant value $cU = p_{11} + p_{22} + p_{33}$;

$p_{ik} = \int_{(4\pi)} J \cos(\widehat{x_i, s}) \cos(\widehat{x_k, s}) \, d\omega$, $(i, k = 1, 2, 3)$ components of tensor P;

$p_{ii} = \int_{(4\pi)} J \cos^2(\widehat{x_i, s}) \, d\omega$, $(i = 1, 2, 3)$ normal components of tensor P;

$X_i = kx_i$, dimensionless co-ordinate along the main axis of the tensor;

τ, time;

E_{eff} and E_{inc}, hemispherical densities of effective and incident radiant fluxes, respectively;

$$\varkappa_{eff} = \frac{\int_{(2\pi)} J_{eff} \, \eta^2 \, d\omega}{\int_{(2\pi)} J_{eff} \, \eta \, d\omega}, \quad \varkappa_{inc} = \frac{\int_{(2\pi)} J_{inc} \, \eta^2 \, d\omega}{\int_{(2\pi)} J_{inc} \, \eta \, d\omega}$$

coefficients allowing for the angular intensity distribution J in the effective and incident fluxes on the boundary surface;

$\eta \equiv \cos\theta$; θ, angle between the beam direction and the normal to the boundary surface;

q_1 and q_2, net radiative fluxes on the boundary surfaces;

q_v, reduced strength of volume heat sources;

$R_1 = 1 - A_1$ and $R_2 = 1 - A_2$, reflection factors of the boundary surfaces;

$K_2(k\delta)$; $K_3(k\delta)$; $K_4(k\delta)$, the second, third and fourth exponential integrals of $k\delta$ obtained from formula

$$K_n(x) = \int_1^\infty \frac{e^{-wx}}{w^{n+1}} \, dw.$$

RADIANT heat transfer processes under steady conditions are described mathematically rigorously by Fredholm integral equations of the second kind. Their analytical solution is difficult especially for systems with volume absorption.

The quest for effective methods of radiant heat transfer prediction has led to the treatment based on the differential equation of radiant transfer. This equation, with no scattering in the medium, is of the form:

$$\frac{1}{k} \frac{dJ}{dl} = -J + J_o. \tag{1}$$

Integrating the transfer equation in all possible directions over 4π radians, the radiant energy balance equation at the point of interest may be obtained,

$$\frac{\partial U}{\partial \tau} + \operatorname{div} \mathbf{q} = 4k\, E_o - kcU = q_v. \tag{2}$$

Stationary problems of radiant transfer only are considered in this paper, problems where the term $\partial U/\partial \tau$ in equation (2) is zero. This equation is, therefore, simplified to:

$$\operatorname{div} \mathbf{q} = 4kE_o - kcU = q_v. \tag{2a}$$

Combination of equations (1) and (2a) yields an integro-differential equation based on the intensity of radiation J. This equation may be solved mathematically using certain assumptions.

Pioneers of the differential methods for radiant heat transfer prediction were, probably, Schuster [1] and Schwarzschild [2]. In 1905–1906 they developed a method which allowed them to study radiative transfer in plane layers of the atmosphere. Their method was used, with some modifications, for geophysical problems by Kuznetsov [3]. Bruhat [4], Nevsky [5], Ribaud [6] and others applied this technique to thermal problems.

The Schuster–Schwarzschild method is based on the conception of the net radiative flux in a plane layer of the medium as the difference between two counter fluxes. This conception allows the integro-differential equation of radiant heat transfer to be approximated by a system of two averaged differential equations for radiative fluxes E_+ and E_-. With no scattering in the medium these equations have the form:

$$\left.\begin{aligned}\frac{dE_+}{dx} &= 2kE_0 - km_+E_+ \\ \frac{dE_-}{dx} &= -2kE_0 + km_-E_-\end{aligned}\right\} \quad (3)$$

The coefficients m_+ and m_- are, generally speaking, functions of the co-ordinate x and other factors. Their behaviour is difficult to predict for every specific case. However, it can be shown by the analysis, m_+ and m_- are slightly dependent on the co-ordinate and they may be substituted in equation (3) for their average value $m_+ = m_- = \bar{m}$. After this substitution joint solution of equations (3) and energy equation (2a) (also in terms of $\bar{m}$) is not difficult in principle. If the fluxes E_+ and E_- are assumed to be perfectly diffusive in every cross section of the layer (which, in general, is close to reality), it is easy to see that $m_+ = m_- = \bar{m} = 2$. For this case solution of intial equations (2a) and (3) yields the following expression for the net radiative flux in a plane layer of the medium bounded by grey walls:

$$q = \frac{E_{0\cdot 1} - E_{0\cdot 2}}{1/A_1 + 1/A_2 - 1 + k\delta}. \quad (4)$$

The Schuster–Schwarzschild method may be valid in principle for the radiant systems of spatial geometry. However, in this case the equations will be more complicated and estimation of m becomes rather difficult.

In 1931 Rosseland [7] presented another differential method for the study of radiation in stellar atmospheres using integral transfer equation (1). In this study Rosseland arrived at a tensor representation of the radiative flux vector:

$$\mathbf{q} = -\frac{1}{k}\operatorname{div} P. \quad (5)$$

Rosseland resolved the radiation tensor P into two parts (the first part is related to the density of radiation energy at a given point, the second gives the intensity distribution over all the directions) and analysed them. His result was that for stellar atmospheres of high optical density the second part of the tensor may be neglected as compared with the first one. This allowed him to obtain the gradient expression for the radiative flux vector:

$$\mathbf{q} = -\frac{c}{3k}\operatorname{grad} U. \quad (6)^*$$

Gradient representations of the radiation vector have been utilized for thermal experiments by Polyak [8] and Shorin [9], who applied this technique to some problems of radiative transport under various conditions, for systems of different geometry.

In 1957 Polyak [10] analysed in detail tensor expression (5) and obtained the following expression for the radiant flux vector by introducing a dimensionless normalized tensor:

$$\mathbf{q} = -\frac{Ac}{k}\operatorname{grad} U - \frac{cU}{k}\operatorname{div} A. \quad (7)$$

Gradient formula (6) may be evidently obtained from equation (7), if the dimensionless tensor A, characterizing the shape of a tensor ellipsoid of the radiation field, is always constant and remains similar to itself. This condition is fulfilled more precisely when the optical density of the medium increases and when the system approaches thermodynamic equilibrium as close as possible. Thus both differential methods (the Schuster–Schwarzschild method and the diffusive one) use approximate differential equations. In view of this, the accuracy of the results obtained by these methods are often open to question. Hottel [11, 12], for instance, criticized the radiative diffusion method. In his work Hottel compared the results of solution of the radiant heat transfer problem in a plane layer of an absorbing medium based on the integral equation† with the solution to the same problem carried out with the radiative diffusion

* Formula (6) is often interpreted in literature as the diffusion approximation of radiant transfer of the similarity with the diffusion equation.

† The integral equation was solved numerically with a computer.

process method. The comparison revealed considerable disagreement between the results. The difference between them is due to the fact that Hottel used for comparison the results obtained by the Konakov approximate method [13]. If the boundary conditions of the problem are formulated more precisely, the difference between the results will be not so striking but may be rather small, as will be demonstrated below.

Besides the two above differential methods tensor expression (5) may be adequately used for some cases of radiative transport in an absorbing medium. For steady conditions relation (5) is rigorous and there is no doubt as to the validity of its use. All the errors encountered are caused only (in contrast to the above methods) by inaccuracy in formulating the boundary conditions.

The solution of a one-dimensional problem of radiant heat transfer is given below on the basis of tensor expression (5), and the results for a plane layer obtained by various differential methods are compared with the numerical solution of Hottel.

From (5) the components of the radiation vector q_{x_i} along the main axes x_1, x_2, x_3 of the tensor P are found from normal components of the tensor p_{ii}:

$$q_{x_i} = -\frac{1}{k}\frac{\partial p_{ii}}{\partial x_i} = -\frac{\partial p_{ii}}{\partial X_i}, \quad i = 1, 2, 3. \qquad (8)$$

Equation (8) is both a refinement of and generalization of the Schuster–Schwarzschild approximation (3):

$$q_x = -\frac{1}{k\bar{m}}\frac{\mathrm{d}(E_+ + E_-)}{\mathrm{d}x} = -\frac{1}{k\bar{m}^2}\frac{\mathrm{d}(cU)}{\mathrm{d}x} \qquad (3a)$$

and of gradient approximation (6)

$$q_x = -\frac{1}{3k}\frac{\partial(cU)}{\partial x}. \qquad (6a)$$

Based on equation (8) energy equation (2a) may be written as:

$$\operatorname{div}\mathbf{q} = \sum_{i=1}^{3}\frac{\partial q_{x_i}}{\partial x_i} = -\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{1}{k}\frac{\partial p_{ii}}{\partial x_i}\right) = k(4E_o - cU) = q_v. \qquad (9)$$

With no heat sources in the medium or with local radiant equilibrium ($q_v = 0$) equation (9) is simplified to:

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{1}{k}\frac{\partial p_{ii}}{\partial x_i}\right) = 0. \qquad (10)$$

For a problem of radiant heat transfer in a plane layer the main axis of tensor $x_i \equiv x$ merges with the direction of the radiative flux vector and is normal to the boundary surfaces. Equation (10) for this case acquires a simple form:

$$\operatorname{div}\mathbf{q} = -\frac{1}{k}\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} = 0 \qquad (11)$$

where

$$p \equiv p_{xx} = \int_{(4\pi)} J\cos^2(\widehat{x, s})\,\mathrm{d}\omega.$$

Boundary conditions may be formulated based on the following considerations.

The net radiative flux q_n normal to the boundary surface may be expressed by two equations:

$$q_n = AE_{\text{inc}} - \epsilon E_o = E_{\text{inc}} - E_{\text{eff}}. \qquad (12)$$

For the one-dimensional problem considered the main tensor axis is normal to the boundary surface. Hence, the value of the normal tensor at the boundary $p_F = p_n$ may be expressed by

$$p_n = \int_{(+2\pi)} J\eta^2\,\mathrm{d}\omega + \int_{(-2\pi)} J\eta^2\,\mathrm{d}\omega = \varkappa_{\text{eff}}E_{\text{eff}} + \varkappa_{\text{inc}}E_{\text{inc}}. \qquad (13)$$

The following expression of the boundary conditions is based on equations (12) and (13):

$$q_n = \frac{1-R}{R\varkappa_{\text{eff}} + \varkappa_{\text{inc}}}p_n - \frac{\varkappa_{\text{eff}} + \varkappa_{\text{inc}}}{R\varkappa_{\text{eff}} + \varkappa_{\text{inc}}}\epsilon E_o. \qquad (14)$$

Solution of energy equation (11) with boundary conditions (14) yields the following formula for the net radiative flux when the temperature of the boundary surfaces is known:

$$q_1 = -q_2 = \frac{(\epsilon_1/A_1)E_{0\cdot 1}(\varkappa_{\text{eff }1} + \varkappa_{\text{inc }1}) - (\epsilon_2/A_2)E_{0\cdot 2}(\varkappa_{\text{eff }2} + \varkappa_{\text{inc }2})}{k\delta + (1/A_1)(\varkappa_{\text{eff }1}R_1 + \varkappa_{\text{inc }1}) + (1/A_2)(\varkappa_{\text{eff }2}R_2 + \varkappa_{\text{inc }2})}. \qquad (15)$$

To use formula (15) one should know the values of coefficients $\varkappa_{\text{eff}}$ and $\varkappa_{\text{inc}}$ on both surfaces. Being obtained from rigorous relation (5), equation (15) is precise, and the results depend only on the accuracy of the coefficients $\varkappa_{\text{eff}}$ and $\varkappa_{\text{inc}}$ given at the boundary. If fluxes E_{eff} and E_{inc} on the boundary surfaces are assumed to be perfectly diffusive (J = const.), these coefficients are

$$\varkappa_{\text{eff}} = \varkappa_{\text{inc}} = \frac{\int_{(2\pi)} J\eta^2\, d\omega}{\int_{(2\pi)} J\eta\, d\omega} = \frac{2\pi \int_0^{\pi/2} \cos^2\theta \sin\theta\, d\theta}{2\pi \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta} = \frac{2}{3}. \qquad (16)$$

In this case formula (15) for a plane layer (at $\epsilon_1 = A_1$ and $\epsilon_2 = A_2$, i.e. with grey walls) becomes a convenient solution of this problem based on gradient formula (6):

$$q = \frac{E_{0\cdot 1} - E_{0\cdot 2}}{(1/A_1) + (1/A_2) - 1 + (3/4)\,k\delta}. \qquad (17)$$

One can see that equation (17) differs from formula (4) obtained by means of the Schuster–Schwarzschild method only by the multiplier 3/4 with $k\delta$ in the dominator. If the boundary surfaces are assumed to be diffusively reflecting, then $\varkappa_{\text{eff}\,1} = \varkappa_{\text{eff}\,2} = (2/3)$ and only the coefficients $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ in equation (15) are to be determined. For the analysis of their values the extreme case of non-equilibrium radiant heat transfer in a plane layer

$$(T_2 = 0,\ \epsilon_1 = \epsilon_2 = A_1 = A_2 = 1)$$

is considered below. In this case the coefficient $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ should differ by the maximum value and depend most strongly on the optical depth of the layer $\Delta = k\delta$.

Formula (15) for this condition

$$(T_2 = 0,\ A_1 = A_2 = 1)$$

reduced to a dimensionless form is simplified to:

$$\left.\frac{q}{E_{0\cdot 1} - E_{0\cdot 2}}\right|_{\substack{T_2=0\\ A_1=A_2=1}} = \frac{q}{E_{0\cdot 1}} = \frac{(2/3) + \varkappa_{\text{inc}\,1}}{\Delta + \varkappa_{\text{inc}\,1} + \varkappa_{\text{inc}\,2}}. \qquad (18)$$

To determine $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ the temperature field in the layer should be known. Uniform temperature distribution over the thickness of the layer was adopted as the first approximation, the temperature being determined from consideration of the problem as a system composed of three zones (two walls and a layer of the medium).

In this case the temperature of the layer is

$$T_G^4 = \frac{T_1^4 + T_2^4}{2} = \frac{T_1^4}{2}, \text{ since } (T_2 = 0) \qquad (19)$$

and the expressions for $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ are the following:

$$\left.\varkappa_{\text{inc}\,1}\right|_{T_2=0} = \frac{2}{3}\,\frac{1 - 3K_3(\Delta)}{1 + 2K_2(\Delta)};$$
$$\left.\varkappa_{\text{inc}\,2}\right|_{T_2=0} = \frac{2}{3}\,\frac{1 + 3K_3(\Delta)}{1 - 2K_2(\Delta)}. \qquad (20)$$

The temperature field was found as the second approximation based on gradient formula (6) with corresponding boundary conditions on the walls. In this case the distribution of the fourth power of T_4 over the layer thickness is linear. Expressions for $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ become more complicated:

$$\left.\varkappa_{\text{inc}\,1}\right|_{T_2=0} = \frac{2}{3}\,\frac{(3/2)\,\Delta - (1/8) + \{(9/8)\,[-4K_4(\Delta)] - 3K_3(\Delta)\}}{(3/2)\,\Delta + [3K_3(\Delta) + 2K_2(\Delta)]} \qquad (21)$$

$$\left.\varkappa_{\text{inc}\,2}\right|_{T_2=0} = \frac{2}{3}\,\frac{(17/8) - \{(9/8)\,[-4K_4(\Delta)] - 3K_3(\Delta)\}}{2 - [3K_3(\Delta) + 2K_2(\Delta)]}. \qquad (22)$$

Thus we can analyse the three solutions of the radiant heat-transfer problem in a plane layer of an absorbing medium which are carried out by various differential methods. Formula (4) corresponds to the solution by the Schuster–Schwarzschild method when assuming that $m_+ = m_- = \bar{m} = 2$ and formula (17) corresponds to the solution by means of gradient expression (6) for radiation vector. Equation (18) is a solution for the particular case ($T_2 = 0$; $A_1 = A_2 = 1$) using tensor relation (5); the values of $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ can be obtained by formulae (20), (21) and (22) for the first and the second approximation.

In Table 1 the results are compiled of dimensionless fluxes between the boundary surfaces of the layer $[q_{\text{rad}}/(E_{0\cdot 1} - E_{0\cdot 2})]$ and

Table 1

$\Delta = k\delta$	Data obtained by the Hottel numerical solution using Fig. 2 of the work [12], $A_1 = A_2 = 1$		Data obtained by the Schuster–Schwarzschild method, formula (4), $A_1 = A_2 = 1$		Data obtained by the method of gradient representations of the radiation vector, formula (17), $A_1 = A_2 = 1$		Data obtained by the proposed method, formula (18) $\varkappa_{inc}$ by (20), $A_1 = A_2 = 1, T_2 = 0$		Data obtained by the proposed method, formula (18), $\varkappa_{inc}$ by (21–22) $A_1 = A_2 = 1, T_2 = 0$	
	$\frac{q_{rad}}{E_{0.1} - E_{0.2}}$	$\frac{q_{G\rightleftarrows S}}{E_{0.1} - E_{0.2}}$	$\frac{q_{rad}}{E_{0.1} - E_{0.2}}$	$\frac{q_{G\leftarrow S}}{E_{0.1} - E_{0.2}}$	$\frac{q_{rad}}{E_{0.1} - E_{0.2}}$	$\frac{q_{G\rightleftarrows S}}{E_{0.1} - E_{0.2}}$	$\frac{q_{rad}}{E_{0.1} - E_{0.2}}$	$\frac{q_{G\rightleftarrows S}}{E_{0.1} - E_{0.2}}$	$\frac{q_{rad}}{E_{0.1} - E_{0.2}}$	$\frac{q_{G\rightleftarrows S}}{E_{0.1} - E_{0.2}}$
0·00	1·0000	0·0000	1·0000	0·0000	1·0000	0·0000	1·0000	0·0000	1·0000	0·0000
0·10	0·9103	0·0777	0·9091	0·0765	0·9302	0·0976	0·9160	0·0834	0·9159	0·0833
0·20	0·8492	0·1453	0·8333	0·1294	0·8696	0·1657	0·8502	0·1463	0·8496	0·1457
0·30	0·7960	0·1959	0·7692	0·1691	0·8163	0·2162	0·7955	0·1954	0·7941	0·1940
0·40	0·7482	0·2336	0·7143	0·1997	0·7692	0·2546	0·7487	0·2341	0·7467	0·2321
0·50	0·7062	0·2630	0·6667	0·2235	0·7273	0·2841	0·7079	0·2647	0·7051	0·2619
0·60	0·6691	0·2860	0·6250	0·2419	0·6897	0·3066	0·6719	0·2888	0·6683	0·2852
0·70			0·5882	0·2561	0·6557	0·3236	0·6396	0·3075	0·6354	0·3033
0·80	0·6045	0·3159	0·5556	0·2670	0·6250	0·3364	0·6106	0·3220	0·6057	0·3171
0·90			0·5263	0·2749	0·5970	0·3456	0·5842	0·3328	0·5789	0·3275
1·00	0·5523	0·3329	0·5000	0·2806	0·5714	0·3520	0·5601	0·3407	0·5543	0·3349
2·00	0·3909	0·3306	0·3333	0·2730	0·4000	0·3397	0·3970	0·3367	0·3905	0·3302
3·00	0·3016	0·2837	0·2500	0·2321	0·3077	0·2898	0·3069	0·2890	0·3019	0·2840
4·00	0·2487	0·2432	0·2000	0·1945	0·2500	0·2445	0·2498	0·2443	0·2461	0·2406
5·00	0·2095	0·2078	0·1667	0·1650	0·2105	0·2088	0·2105	0·2088	0·2078	0·2061
6·00	0·1808	0·1802	0·1429	0·1423	0·1818	0·1812	0·1818	0·1812	0·1798	0·1792
7·00			0·1250	0·1248	0·1600	0·1598	0·1600	0·1598	0·1584	0·1582
8·00	0·1463	0·1462	0·1111	0·1110	0·1429	0·1428	0·1429	0·1428	0·1416	0·1415
9·00			0·1000	0·1000	0·1290	0·1290	0·1290	0·1290	0·1280	0·1280
10·00	0·1209	0·1209	0·0909	0·0909	0·1176	0·1176	0·1177	0·1177	0·1168	0·1168

between the boundary surface and the layer $[q_{G \rightleftarrows S}]/(E_{0\cdot 1} - E_{0\cdot 2})$, obtained by formulae (4), (17) and (18). Here expressions for $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ were taken for the first and the second approximation. However, one can see from Table 1 that the results of both the approximations are in good agreement. In Table 1 for comparison, the values of the same fluxes are shown, as obtained by the Hottel numerical solution using Fig. 2 of his work [12]. Figs. 1 and 2 are

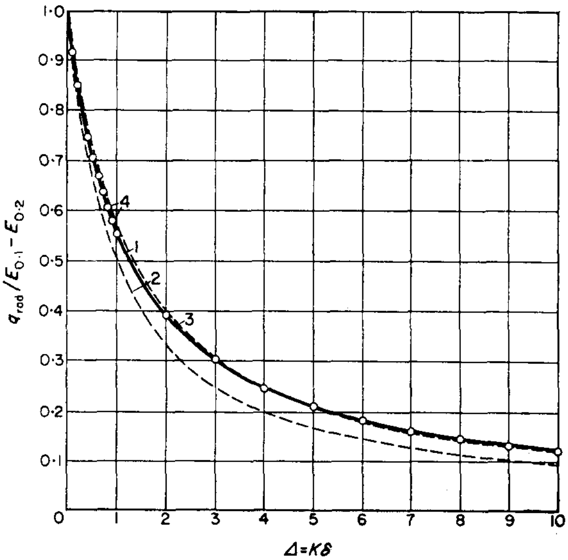

FIG. 1. Effect of the optical depth $\Delta = k\delta$ on dimensionless flux $q_{\text{rad}}/(E_{0\cdot 1} - E_{0\cdot 2})$ through the plane medium between the walls at $A_1 = A_2 = 1$.
Curve 1—the rigorous Hottel solution.
Curve 2—the Schuster–Schwarzschild method, (4).
Curve 3—the method which uses gradient representation of the radiation vector, (17).
4–○—points obtained by the present method, formula (18); $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ are found by formulae (21) and (22) of the second approximation. They merge with curve 1 on the plot.

plotted on the basis of the data of Table 1. Here solutions obtained by all the three differential methods are compared with the numerical solution of Hottel. One can see that the solutions based on the tensor representation of the radiation vector are the most rigorous and exactly coincide with the Hottel numerical solution. Therefore the results obtained by formula (18) which corresponds to the solution by means of

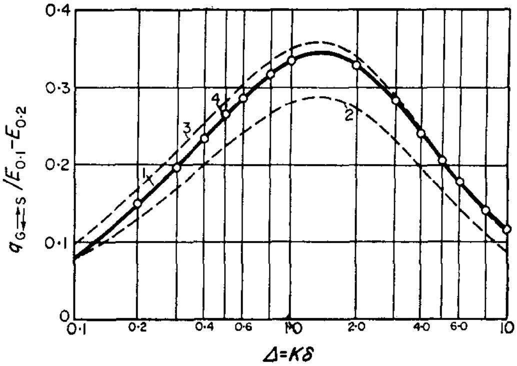

FIG. 2. Effect of the optical depth $\Delta = k\delta$ on the dimensionless radiative flux $q_{G \rightleftarrows S}/(E_{0\cdot 1} - E_{0\cdot 2})$ between the boundary surface (S) and the medium (G) at $A_1 = A_2 = 1$.
Curve 1—the Hottel numerical solution.
Curve 2—the Schuster–Schwarzschild method.
Curve 3—the method which uses gradient representations of the radiation vector.
4–○—points obtained by the present method; $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ are found by formulae (21) and (22) of the second approximation; they merge with curve 1 on the plot.

tensor expression (5) are plotted in Figs. 1 and 2 as separate points actually merging with Hottel's curve. The coefficients $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ for the points plotted are calculated by formulae (21) and (22) of the second approximation. It was already mentioned that the results obtained from formula (18) and the coefficients $\varkappa_{\text{inc}\,1}$ and $\varkappa_{\text{inc}\,2}$ calculated by formula (20) of the first approximation are almost the same as those obtained by the formulae of the second approximation, and plotting these points is, therefore, rather difficult.

The results obtained by the first two differential methods [see (4) and (17)] merge with the rigorous solution only at a certain portion, and over the rest of the optical depth range $k\delta$ they differ to some degree. The solution by the Schuster–Schwarzschild method (assuming that $m_+ = m_- = 2$) merges with that obtained by the Hottel numerical solution over the range of small $k\delta$. Further, with increase in $k\delta$ the curves diverge and at great values of the optical depth expression (4) obtained is 25 per cent lower than the rigorous solution. This is caused by the fact that the assumption of perfect diffusivity ($m_+ = m_- = 2$) with counter fluxes in any cross-section of the layer is satisfied with the greatest accuracy at some values of $k\delta$. The solution by means of gradient expressions of the radiation vector merges with the Hottel solution over the range of great $k\delta$, i.e. in that very place where general tensor expression (5) of the radiation vector may be substituted for gradient formula (6). With decrease in $k\delta$ the solution by gradient formula (6) lies slightly above the rigorous solution and merges with it at $k\delta = 0$. However, the results obtained by all the three differential methods do not differ so markedly from the rigorous solution as claimed by Hottel when he compared the numerical solution with the inaccurate method by Konakov [13]. The inaccuracy of Konakov's method is caused by the fact that he considered it possible to neglect absorption and emission in the layers with optical depth $k\delta < 2$. However, as is shown in the present work, if the boundary conditions are formulated correctly, the differential methods for calculation of radiant heat transfer are satisfactory and may be adequately used for investigation into the problem provided that such incorrect assumptions as Konakov's concept are not made.

CONCLUSIONS

The differential method advocated in this paper for investigation of radiant heat transfer, using tensor representation of radiant flux, is the most accurate one, since it is based on rigorous relations. Calculation carried out by this method agrees well with the Hottel numerical solution which is assumed to be rigorous over the whole range of the optical depth $k\delta$. At small $k\delta$ the Schuster–Schwarzschild method gives good results, and with large values of $k\delta$ the best solution is obtained by gradient expressions of the radiation vector.

The method of radiant heat-transfer study using differential equations (2) and (8) and boundary conditions (12–14) was proposed by Polyak. The mathematical operations for this method were applied to the problems considered in the paper, and all the necessary calculations and illustrations were made by V. N. Adrianov.

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Abstract—This paper describes novel differential methods for studying radiant heat transfer which might have wide application on the score of simplicity. A new method is put forward, based on tensor representation and yields more accurate results. Comparison between calculated results of radiant heat transfer through a plane absorbing layer obtained with existing and with the proposed methods and those obtained numerically on a computer reveal good agreement.

Résumé—Cet article décrit des méthodes différentielles nouvelles pour l'étude des échanges thermiques par rayonnement dont la simplicité permet une large application. Une méthode nouvelle utilisant la représentation tensorielle conduit à des résultats plus précis. Les résultats relatifs aux échanges thermiques par rayonnement dans le cas d'une couche plane absorbante, calculés par les méthodes antérieures, par les méthodes proposées ici et celles qui utilisent des procédé numériques sont en bon accord.

Zusammenfassung—In der Arbeit werden neuartige Differentialmethoden zur Untersuchung des Wärmeübenganges infolge Strahlung beschrieben, die wegen ihrer Einfachheit verbreitete Anwendung finden könnten. Eine neue Methode wird erläutert; Sie basiert auf der Tensorrechnung und führt auf genauere Ergebnisse. Der Vergleich der Rechenergebnisse für den Wärmetransport infolge Strahlung durch eine ebene, absorbierende Schicht nach bereits bekannten und den neu vorgeschlagenen Methoden mit numerisch auf einer Rechenmaschine erhaltenen zeigt gute Übereinstimmung.